COMPSCI/MATH 290-04 Midterm Exam: Spring 2016

NAME: ____

HONOR CODE: I pledge my honor that I will not violate the honor code during this examination

Signature:____

3/10/2016

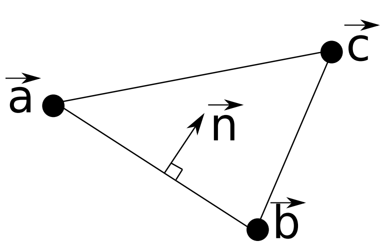
Fill out the answers to the questions below to the best of your ability. You will have 75 minutes to complete this exam. There are 50 total points. Please show your work to maximize partial credit

1 Vectors (8 Points)

1. What is the projection of the vector $\vec{v_1} = (1, -2, 1)$ onto the vector $\vec{v_2} = (2, 1, 2)$?

2. What is the *perpendicular* projection of the vector $\vec{v_1} = (3, 1, 0)$ onto the vector $\vec{v_2} = (1, 4, 1)$? [Hint: It's what's left over after subtracting the projection]

3. Given a triangle in 3D with points $\vec{a}$, $\vec{b}$, and $\vec{c}$, write an equation using vector operations, dot products, and cross products to give an expression for $\vec{n}$, a vector which is normal to the side $\vec{ab}$, which lies in the plane of the triangle, and which points towards the interior of the triangle (see image below for illustration). To save writing, you do not have to normalize this vector. Also, don't use coordinates! Write everything in terms of operations at the vector level



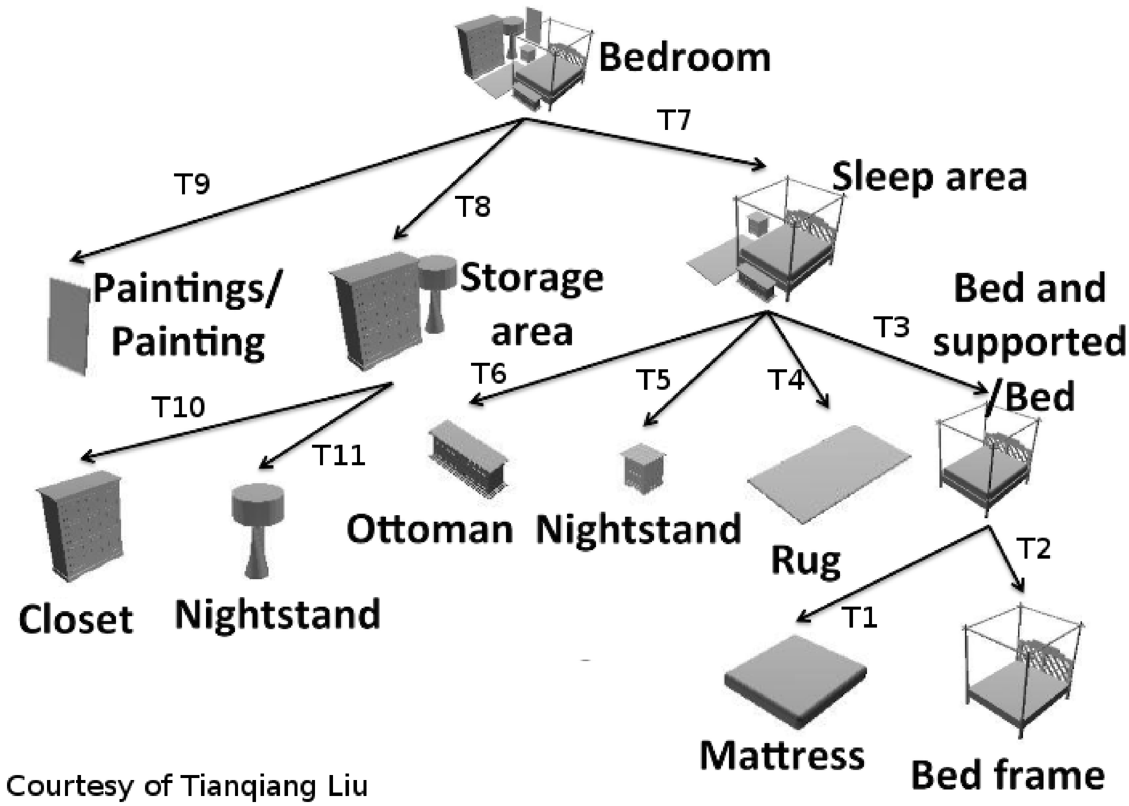
2 Linear Transformations / Matrices (12 Points)

4. Give one reason why it's impossible to represent the translation function $f(x, y) = (a + x, b + y)$, for some constant offset vector $(a, b) \neq (0, 0)$, as a 2×2 matrix. How do we fix this so that we can write a translation in matrix form?

5. Give an example of two 2×2 matrices A and B that don't commute; that is, $AB \neq BA$ (hint: make one of the matrices a shear matrix and there's a good shot the other one won't commute)

6. Given a unit vector $\vec{u}$, is the function *project onto* $\vec{u}$ a linear operation? If so, come up with a matrix A so that Ax performs this projection on a column vector x (you can assume u is a column vector and u^T is the row vector describing $\vec{u}$). If not, show a counter example

7. Recall that there is one degree of freedom in a 2D rotation and three degrees of freedom in a 3D rotation (one for each Euler angle). How many degrees of freedom are there in a rotation of a 4 dimensional Euclidean vector? [Hint: In 3D we broke it down into 3 planes of rotation that were composed of unit axes directions, and there were 3 such planes: xy, yz, and xz. How many unique planes can you make out of 4 directions x, y, z, and w?].
   EXTRA CREDIT: Given a d dimensional Euclidean vector, come up with an expression in terms of d for how many degrees of freedom there are in a rotation of this vector.



3 Scene Graphs (8 Points)

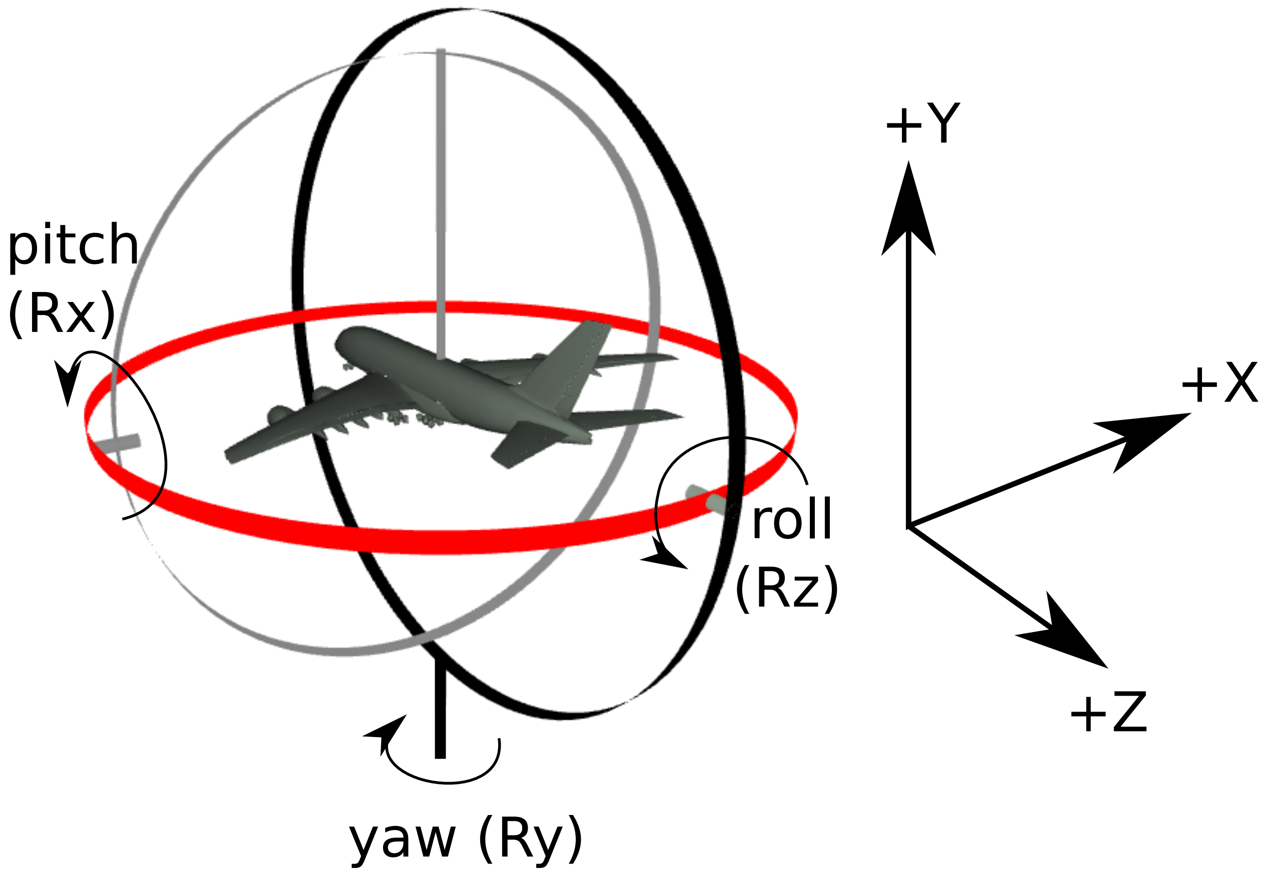
In the scene graph above, each arrow is endowed with a matrix T_x which describes how to transform the object at the head of the arrow into its parent's coordinate system (the same thing as node.transform in group assignment 1). Write a matrix product describing the following transformations, with the convention that a transformation of a column vector x by a matrix A is Ax:

8. The transformation that takes points in the local coordinate system of the closet into world coordinates

9. The transformation that takes points in the local coordinate system of the mattress to the coordinate system of the sleep area

10. The transformation that takes points in the storage area into world coordinates

11. The transformation that takes the paintings into the coordinate frame of the rug [hint: Use the inverse transformation of a matrix is T_x, written as T_x^{-1}. T_x^{-1} undoes the original transformation so that $T_x T_x^{-1}$ and $T_x^{-1} T_x$ are the identity]



4 Euler Angles / Gimbal Lock (8 Points)

The above figure shows a gimbal system in which the pitch ring (gray) is the inner ring, the roll ring (red) is the middle ring, and the yaw ring (black) is the outer ring. The pitch ring is connected to the roll ring, and the roll ring is connected to the yaw ring. The airplane is connected to the pitch ring.

12. Given that a canonical pitch rotation is the rotation matrix about X, R_X, the canonical yaw rotation is a rotation about Y, R_Y, and the canonical roll rotation is a rotation about Z, R_Z, write the matrix product which composes all canonical rotations to represent the global motion of the airplane.

13. Which two rings have to align for gimbal lock to occur? When these two rings are aligned, which two rings have the same effect (possibly in opposite directions)?

5 Eigenvalues/Eigenvectors (6 Points)

14. Circle all of the vectors below which are eigenvectors of the matrix $\begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}$. For each eigenvector you circle, also write the associated eigenvalue below it

$$\begin{bmatrix} -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

15. Show that $\begin{bmatrix} 1 \\ i \end{bmatrix}$ (where $i = \sqrt{-1}$) is an eigenvector of the 2D rotation matrix that rotates a vector by θ in the counter-clockwise direction. What is the eigenvalue associated with this eigenvector? [Hint: Use Euler's formula]

6 Quaternions (4 Points)

16. Given a point $\vec{P} = (a, b, c)$, write the product of three quaternions that performs a rotation around the unit axis $\vec{u} = (d, e, f)$ ($||\vec{u}|| = 1$), by an angle θ. That is, if you put P in a quaternion $ai + bj + ck$, then what do you need to multiply P by on either side so that the imaginary components of the resulting multiplication hold the rotated point? No need to expand the multiplication, just write what the quaternions are.

7 PCA (4 Points)

17. In the space below, sketch two 2D point clouds. Make the first point cloud such that the PCA axis magnitudes very close to each other. Make the second point cloud such that they are very different. What is the danger in the first case if we're going to use these axes to represent the orientation of this shape for comparison with other similar shapes?

8 Duality (Extra Credit: +5 Points)

18. Given a set of line segments in the plane, the "stabbing lines" problem is to determine whether or not there exists a line that intersects all of them (see picture below). Checking to see if a line stabs a set of line segments is equivalent to what problem in the dual? [Hint: A stabbing line has to be below all of the upper points on the line segments and above all of the lower points on the line segments]